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New relations between the Clebsch–Gordan coefficients of SU(2)

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Abstract. Some unknown bilinear relations between the SU(2) Clebsch–Gordan coefficients are given. The derivation is based on the analysis of the triplet $\mathrm{P} \supset \mathrm{L} \supset \mathrm{SO}(3)$ (Poincaré, Lorentz and rotation group). The form of the covariant p_μ-dependent functions with fixed maximal degree with respect to p_μ is given.

1. Introduction

In this paper we derive some unknown bilinear relations between the Clebsch–Gordan coefficients of the SU(2) group. These relations can be applied to the analysis of the free field equations (Rembieliński 1980) and especially to the construction of the covariant p_μ-dependent functions with fixed maximal degree with respect to the four-momentum p_μ. The derivation is based on the notion of the commutant $\boldsymbol{X}_p$ of the representation $D(R_p) = D(\Lambda)\downarrow R_p$. Here $D(\Lambda)$ is a (in general, reducible) representation of the Lorentz group $L(\Lambda)$ while R_p is the little group of the four-momentum p_μ with $p^2 > 0$. As is well known, in this case R_p is isomorphic to $\mathrm{SO}(3) \sim \mathrm{SU}(2)$. The elements $X(p)$ of the commutant $\boldsymbol{X}_p = \{X(p)\}$ are defined by the equation

$$X(\Lambda p) = D(\Lambda) X(p) D(\Lambda^{-1}) \tag{1}$$

i.e. for $\Lambda \in R_p$

$$[X(p), D(R_p)] = 0.$$

From Weyl's theorem (Weyl 1939) it follows that the set $\boldsymbol{X}_p$ forms the associative algebra which is the direct sum of the mutually orthogonal subalgebras $\boldsymbol{X}_p^s$ according to the decomposition $D(R) = \bigoplus_s N_s \mathscr{D}^s(R)$. Here s and N_s denote spin and multiplicity of the irreducible representation $\mathscr{D}^s$ of the SU(2) in D. Moreover, in each subalgebra $\boldsymbol{X}_p^s$ there exists a basis $\{X_{i,k}^s(p)\}$ with the multiplication law

$$X_{i,k}^s(p) X_{j,l}^{s'}(p) = \delta^{s,s'} \delta_{k,j} X_{i,l}^s(p). \tag{2}$$

Here the indices i, k, j, l denote equivalent irreducible representations of the R_p with fixed s. The operators $X_{i,k}^s$ intertwine these representations. The irreducible subspaces of R_p can be numbered by triplet $[(A, B)\alpha]$ where α distinguishes the equivalent representations D^{AB} of the Lorentz group (A, B integer or half-integer) contained in D, i.e.

$$X_{i,k}^s(p) \equiv X_{[(A,B)\alpha],[(A',B')\alpha']}^s(p). \tag{3}$$

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This follows from the fact that the representations $\mathscr{D}^s$ occur in D^{AB} with multiplicity one. Note that the base elements (3) are homogeneous with respect to p_μ with homogeneity degree zero (see equation (2)), i.e. they are dimensionless.

The degree with respect to p_μ (not homogeneity degree!) of the dimensionless operator $X(p)$ will be denoted below by $r(X)$. For example

$$r(g_{\mu\nu}) = r(\gamma_\mu) = 0 \qquad r\left(\frac{p_\mu}{\sqrt{p^2}}\right) = r\left(1 + \frac{p\gamma}{\sqrt{p^2}}\right) = 1 \qquad r\left(\delta^\nu_\mu - \frac{p_\mu p^\nu}{p^2}\right) = 2 \qquad \text{etc.}$$

It is easy to see that the following rules hold:

$$r(X_a X_b) \leqslant r(X_a) + r(X_b) \tag{4a}$$

$$r(X_a + X_b) \leqslant \max\{r(X_a), r(X_b)\} \tag{4b}$$

$$r(MX) = r(X) \tag{4c}$$

if $r(M) = 0$ and M is invertible;

$$r(X(p')) = r(X(p)) \tag{4d}$$

where $p' = (p_0, \boldsymbol{Rp})$, $R \in \mathrm{SO}(3)$.

2. The basic formulae

In this and the next section we present a sketch of the derivation of the above-mentioned relations between the Clebsch–Gordan coefficients. To avoid misunderstanding we recall that the SU(2) Clebsch–Gordan coefficients are denoted by $\langle ABab|sm\rangle$ where $a = -A, -A+1, \ldots, A$, $b = -B, -B+1, \ldots, B$, $m = a + b$, $|A - B| \leqslant s \leqslant (A + B)$; in the other cases $\langle ABab|sm\rangle = 0$. We adopt the real phase convention (Edmonds 1957).

Let us consider a dimensionless operator $X(p) \in \boldsymbol{X}_p$. It can be expanded in the base $\{X^s_{[(A,B)\alpha],[(A',B')\alpha']}(p)\}$ as follows:

$$X(p) = \sum_{[(A,B)\alpha],[(A',B')\alpha']} \left(\sum_s \omega^s_{[(A,B)\alpha],[(A',B')\alpha']} X^s_{[(A,B)\alpha],[(A',B')\alpha']}(p)\right) \tag{5}$$

where the coefficients $\omega^s_{[(A,B)\alpha],[(A',B')\alpha']}$ are p_μ-independent. On the other hand, denoting by $\Pi^{(A,B)}_\alpha$ the projectors on the irreducible representations D^{AB}, we can rewrite $X(p)$ in the form

$$X(p) = \sum_{[(A,B)\alpha],[(A',B')\alpha']} \Pi^{(A,B)}_\alpha X(p) \Pi^{(A',B')}_{\alpha'}. \tag{6}$$

In the following we will investigate the condition

$$r(X(p)) \leqslant l \tag{7}$$

where $l \geqslant 0$ is an integer. Because $r(\Pi^{(A,B)}_\alpha) = 0$ then from equations (4a), (5) and (6) we see that the condition (7) is satisfied if and only if the relations

$$r\left(\sum_s \omega^s_{[(A,B)\alpha],[(A',B')\alpha']} X^s_{[(A,B)\alpha],[(A',B')\alpha']}(p)\right) \leqslant l \tag{8}$$

hold for all $[(A, B)\alpha]$ and $[(A', B')\alpha']$ (s varies from $\max(|A - B|, |A' - B'|)$ to $\min((A + B), (A' + B'))$).

Because of the relation (4d), we can rotate in $X(p)$ the argument p_μ in the direction of the z axis, i.e. $(p_0; \boldsymbol{p}) \to (p_0; 0, 0, |\boldsymbol{p}|) \equiv q_\mu$. Next we calculate the matrix elements of the operator $X^s_{[(A,B)\alpha],[(A',B')\alpha']}(q)$; we omit the inessential indices α and α'. We denote by $|ABab\rangle$ the (standard) base vectors which span the representation space of D^{AB} ($a = -A, -A+1, \ldots, A, b = -B, -B+1, \ldots, B$). We obtain

$$\begin{aligned}\langle ABab|X^s_{[A_1,B_1],[A_2,B_2]}(q)|A'B'a'b'\rangle &= \langle ABab|D(\Lambda_q)X^s_{[A_1,B_1],[A_2,B_2]}(k)D(\Lambda_q^{-1})|A'B'a'b'\rangle \\ &= \langle ABab|X^s_{[A_1,B_1],[A_2,B_2]}(k)|A'B'a'b'\rangle \exp\{[(a'-a)-(b'-b)]\theta\}. \qquad (9)\end{aligned}$$

Here $k = \Lambda_q^{-1}q = (\sqrt{p^2}; 0, 0, 0)$ and the boost Λ_q is represented by

$$D(\Lambda_q) = \exp[-\mathrm{i}(\boldsymbol{q}/|\boldsymbol{p}|)\boldsymbol{K}\theta] = \exp(-\mathrm{i}\theta K_3)$$

where $\sinh\theta = |\boldsymbol{p}|/\sqrt{p^2}$ and K_i are the generators of the Lorentz boosts.

Note that the $X^s_{[A,B],[A',B']}(k)$ are p_μ-independent because they intertwine equivalent irreducible representations of the static (rest frame) SO(3). Consequently the operator $X^s_{[A_1,B_1],[A_2,B_2]}(k)$ has the form

$$X^s_{[A_1,B_1],[A_2,B_2]}(k) = \sum_{m=-s}^{s} {}_1|sm\rangle\langle sm|_2 \qquad (10)$$

where $|sm\rangle$ are the eigenvectors of the J_3 and $\boldsymbol{J}^2$. From equations (9) and (10) we have

$$\begin{aligned}\langle ABab|X^s_{[A_1,B_1],[A_2,B_2]}(q)|A'B'a'b'\rangle &= \delta_{A,A_1}\delta_{B,B_1}\delta_{A',A_2}\delta_{B',B_2}\left(\frac{p_0+|\boldsymbol{p}|\epsilon(a'-a)}{\sqrt{p^2}}\right)^{2|a-a'|} \\ &\quad\times\left(\sum_{m=-s}^{s}\langle ABab|sm\rangle\langle A'B'a'b'|sm\rangle\right) \qquad (11)\end{aligned}$$

where $\epsilon(x)$ is the sign function. Therefore the conditions (8) are equivalent to

$$\sum_s \sum_{m=-s}^{s} \omega^s_{[A,B],[A',B']}\langle ABab|sm\rangle\langle A'B'a'b'|sm\rangle = 0 \qquad (12)$$

for a and a' satisfying the inequality $2|a-a'| > l$. From equation (11) it follows that the maximal degree of the operator

$$X_{[A,B],[A',B']}(p) = \sum_s \omega^s_{[A,B],[A',B']}X^s_{[A,B],[A',B']}(p)$$

introduced in equation (8) (we omit the indices α, α') is equal to

$$l_{\max} = 2\min((A+A'), (B+B')). \qquad (13)$$

Because for fixed $[A, B]$ and $[A', B']$ the product $2|a-a'|$ in equation (11) is only odd or only even, then the minimal degree $l_{\min}$ is given by

$$l_{\min} = 2\max(|A-A'|, |B-B'|). \qquad (14)$$

Note that

$$\begin{aligned}\tfrac{1}{2}(l_{\max}-l_{\min}) &= \min((A+A'), (B+B')) - \max(|A-A'|, |B-B'|) \\ &= \min((A+B), (A'+B')) - \max(|A-B|, |A'-B'|).\end{aligned}$$

3. The derivation

Let us consider the case when in the inequality (8) l acquires its lowest value, namely

$$r(X^{\min}_{[A,B],[A',B']}) = l_{\min} \tag{15}$$

where

$$X^{\min}_{[A,B],[A',B']} \doteq \sum_s \omega^s_{[A,B],[A',B']_{\min}} X^s_{[A,B],[A',B']}$$

and, as before, s varies from $\max(|A-B|, |A'-B'|)$ to $\min((A+B), (A'+B'))$. As is well known (see, for example, Barut and Rączka 1977) the relativistic spin operator $\hat{S}^2_{(A,B)}$ has the form

$$\hat{S}^2_{(A,B)} = \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} - S_{\mu\sigma} S^{\nu\sigma} \frac{p^\mu p_\nu}{p^2}$$

where $S_{\mu\nu}$ are the Lorentz generators in the representation D^{AB}. Consequently

$$r(\hat{S}^2_{(A,B)}) = 2. \tag{16}$$

Because

$$\hat{S}^2_{(A,B)} = \sum_{s=|A-B|}^{A+B} s(s+1) X^s_{[A,B],[A,B]}$$

then for $X^{\min}_{[A,B],[A',B']}$ satisfying equation (15) the operators

$$X_k \doteq (\hat{S}^2_{(A,B)})^k X^{\min}_{[A,B],[A',B']} = \sum_s [s(s+1)]^k \omega^s_{[A,B],[A',B']_{\min}} X^s_{[A,B],[A',B']}, \tag{17}$$

where $k = 0, 1, \ldots, \frac{1}{2}(l_{\max} - l_{\min})$, are linearly independent. To prove this we note that there must exist a set of the linearly independent operators $X_0, X_1, \ldots, X_t$ of the form (17). If X_{t+1} is linearly dependent on $X_0, X_1, \ldots, X_t$, then it is easy to see that every X_{t+n} for $n = 1, 2, 3, \ldots$ is linearly dependent too. If we take this fact into account, then from the form of the transformation matrix

$$T \doteq [T^s_k \equiv [s(s+1)]^k \omega^s_{[A,B],[A',B']_{\min}}]$$

it follows that the number of non-vanishing coefficients $\omega^s_{[A,B],[A',B']_{\min}}$ is equal to $t+1$. Therefore T is a square matrix and must be invertible. So we can expand the intertwining operators $X^s_{[A,B],[A',B']}$ in terms of the X_k, $k = 0, 1, \ldots, t$, i.e.

$$X^s_{[A,B],[A',B']} = \sum_{k=0}^{t} \alpha^s_k X_k.$$

Then from equations (4*a*, *b*), (15), (16) and (17) we have

$$r(X^s_{[A,B],[A',B']}) = r\left(\sum_{k=0}^{t} \alpha^s_k X_k\right) \leqslant \max_k \{r(X_k)\} \leqslant 2t + l_{\min}.$$

But from equation (11) it follows that $r(X^s_{[A,B],[A',B']}) = l_{\max}$. Therefore

$$t = \tfrac{1}{2}(l_{\max} - l_{\min}) \tag{18}$$

because the number of base operators $X^s_{[A,B],[A',B']}$ is equal to $1 + \frac{1}{2}(l_{\max} - l_{\min})$.

Note that the linear independence of the X_k for $k=0, 1, \ldots, \frac{1}{2}(l_{\max}-l_{\min})$ implies that

$$r((\hat{S}^2)^k X^{\min}_{[A,B],[A',B']}) = 2k + l_{\min}. \tag{19}$$

Consequently, the most general form of the operator satisfying the inequality (8) is the following:

$$X_{[A,B],[A',B']} = \sum_{k=0}^{\frac{1}{2}(l-l_{\min})} \lambda^k_{[A,B],[A',B']} X_k \tag{20a}$$

i.e. from equations (8) and (17) we have

$$\omega^s_{[A,B],[A',B']} = \sum_{k=0}^{\frac{1}{2}(l-l_{\min})} \lambda^k_{[A,B],[A',B']} [s(s+1)]^k \omega^s_{[A,B],[A',B']_{\min}}. \tag{20b}$$

Here the coefficients λ^k are s-independent.

Now we determine the coefficients $\omega^s_{[A,B],[A',B']_{\min}}$. To do this let us consider the product $X^{\min}_{[A,B],[A',B']} X^{\min}_{[A',B'],[A,B]}$ where

$$X^{\min}_{[A'B'],[A,B]} \doteq \sum_s \omega^s_{[A,B],[A',B']_{\min}} X^s_{[A',B'],[A,B]}.$$

Because† $r(X^{\min}_{[A',B'],[A,B]}) = r(X^{\min}_{[A,B],[A',B']}) = l_{\min}$, then from equation (4$a$)

$$r(X^{\min}_{[A,B],[A',B']} X^{\min}_{[A',B'],[A,B]}) \leqslant 2l_{\min}.$$

Using the multiplication rules (2) as well as equations (19) and (20a) we obtain‡

$$\begin{aligned} X^{\min}_{[A,B],[A',B']} X^{\min}_{[A',B'],[A,B]} &= \sum_s (\omega^s_{[A,B],[A',B']_{\min}})^2 X^s_{[A,B],[A,B]} \\ &\equiv \sum_{k=0}^{l_{\min}} \alpha_k (\hat{S}^2_{(A,B)})^k = \sum_{k=0}^{l_{\min}} \alpha_k \sum_{s=\lvert A-B\rvert}^{A+B} [s(s+1)]^k X^s_{[A,B],[A,B]} \end{aligned} \tag{21}$$

where α_k are undetermined constants. Similarly we get the analogous formula for the product $X^{\min}_{[A',B'],[A,B]} X^{\min}_{[A,B],[A',B']}$. Furthermore, if we restrict ourselves for simplicity to the special case $A \geqslant B$, $A' \geqslant B'$, $(A+B) \geqslant (A'+B')$ and $(A-B) \geqslant (A'-B')$ then from these formulae we obtain

$$(\omega^s_{[A,B],[A',B']_{\min}})^2 = \sum_{k=0}^{2(A-A')} \alpha_k [s(s+1)]^k \tag{22}$$

for $(A-B) \leqslant s \leqslant (A'+B')$ and

$$\sum_{k=0}^{2(A-A')} \alpha_k [s(s+1)]^k = 0 \tag{23}$$

† As is well known, under the parity transformation $D^{AB} \to D^{BA}$, i.e. $X^s_{[A,B],[A',B']} \to X^s_{[B,A],[B',A']}$. Moreover, under the Dirac conjugation $\overline{D^{AB}}(\Lambda) = D^{BA}(\Lambda^{-1})$ and consequently $\overline{X^s_{[A,B],[A',B']}} = X^s_{[B',A'],[B,A]}$ (for details see Rembieliński 1980). On the other hand, the coefficients $\omega^s_{[A,B],[A',B']_{\min}}$ can be chosen real. Therefore $X^{\min}_{[A',B'],[A,B]}$ can be obtained by the parity transformation and Dirac conjugation from $X^{\min}_{[A,B],[A',B']}$.

‡ Because $X^{\min}_{[A,B],[A,B]} \sim \Pi^{(A,B)}$, equation (19) implies that $r((\hat{S}^2_{(A,B)})^k) = 2k$ and therefore $r((\hat{S}^2_{(A,B)})^k) \leqslant 2l_{\min}$ for $k=0, 1, \ldots, l_{\min}$. From equation (20a) we see that $\sum_{k=0}^{l_{\min}} \alpha_k (\hat{S}^2_{(A,B)})^k$ is the most general form of the operator with degree less or equal to $2l_{\min}$.

for $(A'+B')<s\leqslant(A+B)$ and $(A'-B')\leqslant s<(A-B)$. The equations (23) determine $2(A-A')$ coefficients α_k (we can choose $\alpha_0=1$) and it is immediately shown that the solutions of the equations (21)–(23) have the form

$$\omega^{s}_{[A,B],[A',B']_{\min}}=\prod_{k=0}^{N-1}[(s_m-k)(s_m-k+1)-s(s+1)]^{1/2}$$
$$\times\prod_{l=0}^{M-1}[s(s+1)-(s_0+l)(s_0+l+1)]^{1/2} \tag{24}$$

where $N=|(A+B)-(A'+B')|$, $M=||A-B|-|A'-B'||$, $s_m=\max((A+B),(A'+B'))$, $s_0=\min(|A-B|,|A'-B'|)$.

Thus from equations (12) and (24) we obtain that for every a and a' obeying $|a-a'|>\frac{1}{2}l_{\min}$ the following relations hold:

$$\sum_{s}\sum_{m=-s}^{s}\left\{\prod_{k=0}^{N-1}[(s_m-k)(s_m-k+1)-s(s+1)]^{1/2}\right.$$
$$\left.\times\prod_{l=0}^{M-1}[s(s+1)-(s_0+l)(s_0+l+1)]^{1/2}\langle ABab|sm\rangle\langle A'B'a'b'|sm\rangle\right\}=0.$$

More general formulae can be obtained with use of the equations (20). They are listed below.

(i) If $A\geqslant B$ and $A'\geqslant B'$ or $A\leqslant B$ and $A'\leqslant B'$ then for each pair a, a' obeying

$$|a-a'|>n_0+n \tag{25}$$

where $n_0=\max\{|A-A'|,|B-B'|\}=\frac{1}{2}l_{\min}$ and $n=0,1,2,3,\ldots$, the following formula is valid:

$$\sum_{s}\sum_{m=-s}^{s}\left\{[s(s+1)]^{n}\prod_{k=0}^{N-1}[(s_m-k)(s_m-k+1)-s(s+1)]^{1/2}\right.$$
$$\left.\times\prod_{l=0}^{M-1}[s(s+1)-(s_0+l)(s_0+l+1)]^{1/2}\langle ABab|sm\rangle\langle A'B'a'b'|sm\rangle\right\}=0. \tag{26}$$

(ii) If $A>B$ and $A'<B'$ or $A<B$ and $A'>B'$ then for each pair a, a' obeying the inequality (25) we have (a) for $A+B$ and $A'+B'$ both integer:

$$\sum_{s}\sum_{m=-s}^{s}\left\{[s(s+1)]^{n}\prod_{i=0}^{s_0-1}[s(s+1)-(s_0-i)(s_0-i-1)]\right.$$
$$\times\prod_{k=0}^{N-1}[(s_m-k)(s_m-k+1)-s(s+1)]^{1/2}$$
$$\left.\times\prod_{l=0}^{M-1}[s(s+1)-(s_0+l)(s_0+l+1)]^{1/2}\langle ABab|sm\rangle\langle A'B'a'b'|sm\rangle\right\}=0 \tag{27}$$

(b) for both $A+B$ and $A'+B'$ half-integer:

$$\sum_{s}\sum_{m=-s}^{s}\left\{(2s+1)[s(s+1)]^{n}\prod_{i=0}^{s_0-\frac{3}{2}}[s(s+1)-(s_0-i)(s_0-i-1)]\right.$$
$$\times\prod_{k=0}^{N-1}[(s_m-k)(s_m-k+1)-s(s+1)]^{1/2}$$
$$\left.\times\prod_{l=0}^{M-1}[s(s+1)-(s_0+l)(s_0+l+1)]^{1/2}\langle ABab|sm\rangle\langle A'B'a'b'|sm\rangle\right\}=0. \qquad (28)$$

Note that for $A=A'$, $B=B'$, $n=0$ we get from (26) the orthogonality relation.

Finally we remark that the equation (1) in fact defines the p_μ-dependent, Lorentz covariant functions. For this reason the equations (5), (7), (12), (26), (27) and (28) determine the form of such functions (also with homogeneity degree different from zero) in terms of the base elements of the commutant $\boldsymbol{X}_p$. These base operators can be determined in each fixed case (for details see Rembieliński 1980).

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